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Real-gas effects in very weak shock waves in the atmosphere and the structure of sonic bangs

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An approximate expression is given for the thickness of weak fully dispersed shock waves. Using available data on the thermodynamic properties of air, it is shown that shocks of the strength expected in sonic bangs are fully dispersed. Estimated relaxation times for dry and humid air lead to wide variations in possible thickness, varying from millimetres to metres.

It is well known that a shock wave in a gas with slow internal modes of energy is fully dispersed if the wave speed lies between the equilibrium and frozen sound speeds. Such waves have on the whole been treated as interesting phenomena of no great practical importance and the aerodynamicists have tended to ignore real-gas effects in air at moderate speeds and temperature and looked at them as essentially high-speed and high-temperature effects. The acousticians, on the other hand, have looked at non-equilibrium effects in sound waves as being essentially frequency-dependent dispersion and attenuation effects.

It is the purpose of this note to point out that the non-ideal gas effects in real air may be sufficiently large to make the sonic-bang shock waves fully dispersed and their thickness several orders of magnitude greater than predicted by the classical theories of shock wave structure. It should be emphasized that the present note is essentially concerned with the structure of a single shock wave and that the implications for the structure of the N-wave system reflected from the ground are necessarily tentative at this stage. We hope to make the point, however, that non-equilibrium effects cannot necessarily be ignored in discussions of the detailed structure of sonic bangs.

Consider a fully dispersed wave in a co-ordinate system fixed in the wave and with the x axis in the direction of flow, and let suffixes 1 and 2 refer to conditions far upstream and far downstream of the wave, respectively. If it is assumed that the rotational and translational modes of energy are always in mutual equilibrium and that normal viscosity and heat-transfer effects are negligible, the conservation equations are in the usual notation

$$\rho u = \rho_1 u_1, \tag{1}$$

$$p + \rho u^2 = p_1 + \rho_1 u_1^2, \tag{2}$$

$$c_p T + \sigma + \tfrac{1}{2}u^2 = c_p T_1 + \sigma_1 + \tfrac{1}{2}u_1^2. \tag{3}$$

Here c_p is the frozen specific heat at constant pressure and σ is the vibrational contribution to the specific internal energy.

The rate equation is

$$d\sigma/dt = \rho\phi(\bar{\sigma}-\sigma), \tag{4}$$

where $\bar{\sigma}$ is the local equilibrium value of the vibrational energy and $\rho\phi$ is the relaxation frequency (inverse relaxation time). The equation of state is

$$p = \rho RT \tag{5}$$

and the equilibrium value of the vibrational energy may be written as

$$\bar{\sigma} = \sigma_1 + cR(T-T_1), \tag{6}$$

where cR is the difference between equilibrium and frozen specific heats. Because the temperature variation in fully dispersed waves in the atmosphere is small (only a few hundredths of a °K) it is permissible to take ϕ and c as constants equal to their initial values ϕ_1 and c_1.

The system of equations (1) to (6) can be solved for any of the six variables p, ρ, u, T, σ, and $\bar{\sigma}$, provided the wave is fully dispersed and hence without discontinuities. The precise condition for this is that the frozen Mach number M_1 lies in the interval given by

$$\frac{1+(\gamma-1)c_1/\gamma}{1+(\gamma-1)c_1} \leqslant M_1^2 \leqslant 1,$$

where γ is the frozen value of the specific heat ratio. The lower limit corresponds to the equilibrium Mach number $M_{e1} = 1$ and hence gives an infinitely weak wave. The upper limit corresponds to the strongest possible fully dispersed wave.

Using the fact that $d/dt = ud/dx$ we get, after some straightforward algebra, the differential equation describing the velocity distribution through the wave

$$\frac{\rho_1\phi_1}{u_1}\frac{dx}{dV} = \frac{2V^2[(\gamma+1)M_1^2V-(1+\gamma M_1^2)]}{M_1^2[(\gamma+1)+2(\gamma-1)c_1][V_1-V][V-V_2]}, \tag{7}$$

where $V = u/u_1$. Equation (7) was essentially derived by Lighthill (1956, equation (232)), and can of course be integrated to find the structure of the wave as done approximately by Lighthill (1956, equation (237)). However, in this note we are mainly interested in finding an overall measure for the thickness of the wave.

It is normal practice to define this thickness in terms of the maximum slope as

$$\delta = (V_2-V_1)(dx/dV)_{\max}.$$

However, in fully dispersed waves it is found that the inflexion point moves from the centre of the wave for an infinitely weak wave to the front of the wave for a fully dispersed wave of maximum strength (wave speed equal to frozen speed of sound).

The standard definition of thickness does not therefore give representative values. Instead we use the definition

$$\Delta = (V_2-V_1)(dx/dV)_{V=\frac{1}{2}(V_1+V_2)}. \tag{8}$$

The thickness thus defined is always comparable with a boundary-layer type thickness, say from $V_1-V = 0{\cdot}1(V_1-V_2)$ to $V_1-V = 0{\cdot}9(V_1-V_2)$.

A further considerable amount of simple algebra gives

$$= \frac{4u_1c_1(1+\gamma M_1^2)^3\,(1+(\gamma-1)\,c_1/\gamma)^2\,(\gamma-1)^2}{\rho_1\phi_1\gamma[(\gamma+1)+2(\gamma-1)\,c_1]^3\,M_1^4[(1+(\gamma-1)\,c_1)\,M_1^2-(1+(\gamma-1)\,c_1/\gamma)]}. \tag{9}$$

For the case of the atmosphere $c_1 \ll 1$ and $1-M_1^2 \ll 1$ and we get the following approximate expression

$$\Delta = \frac{4u_1c_1(\gamma-1)^2}{\rho_1\phi_1\gamma(M_{e1}^2-1)}, \tag{10}$$

where subscript e refers to equilibrium and we have made use of the fact that

$$\frac{1+(\gamma-1)\,c_1}{1+(\gamma-1)\,c_1/\gamma}M_1^2 = \frac{\gamma}{\gamma_e}M_1^2 = M_{e1}^2.$$

Now
$$\frac{p_2-p_1}{p_1} = \frac{2\gamma_e}{\gamma_e+1}(M_{e1}^2-1), \tag{11}$$

where $2\gamma_e/(\gamma_e+1)$ can be put equal to $2\gamma/(\gamma+1)$. With $\gamma = \frac{7}{5}$ we finally get the approximate expression

$$\Delta = \frac{8u_1p_1c_1}{15\rho_1\phi_1(p_2-p_1)}. \tag{12}$$

It is not a difficult matter to calculate the contributions to c_1 for each of the constituents of the atmosphere. The harmonic oscillator model of the vibrational energy modes may be used with confidence. It is found that c_1 is dominated by the oxygen contribution which is at least five times that from either nitrogen or water vapour in the temperature range of interest. All the other constituents make a negligible contribution owing to their minute concentrations.

It was pointed out by Knudsen (1933, 1935) that the absorption of sound in air was due largely to the presence of oxygen. He found that the relaxation frequency of oxygen depended very strongly on the water vapour content. Tuesday & Boudart (1955) showed that the resonance between the vibrational energy levels of oxygen and those of the ν_2 mode of water vapour was sufficient to explain this strong dependence. Their theory and experiments demonstrated that the relaxation frequency of humid oxygen was a quadratic function of the mole fraction of water vapour.

Henderson & Herzfeld (1965) reviewed data on sound absorption in humid air and oxygen. They showed that the effect of nitrogen on the relaxation frequency of oxygen in air could not be ignored.

Further experiments have been made by Monk (1969) over a wider range of water vapour concentration. He concluded that Henderson & Herzfeld overestimated the relaxation frequency in air for mole fractions of water vapour greater than about 0·002. Monk gave an empirical expression based on a modified form of Henderson & Herzfeld's theory for the frequency of maximum sound absorption in humid air due to the vibrational relaxation of oxygen. He claims reasonable accuracy over almost the complete humidity range but recommends Henderson & Herzfeld's quadratic (obtained from experiments by Harris (1963)) for mole fractions less than 0·002.

Since the relaxation frequency of nitrogen is an order of magnitude lower than that of oxygen, and since nitrogen is less affected by water vapour (Piercy 1969), we assume here for simplicity that the internal energy of nitrogen remains frozen throughout the wave. The width of the wave as defined by equation (8) would not be much affected by the relaxation of nitrogen but the width as defined by the separation of the two points $0\cdot1(V_1 - V_2)$ and $0\cdot9(V_1 - V_2)$ might be considerably larger.

Water vapour has a very high relaxation frequency (according to Henderson & Herzfeld) and is assumed to be everywhere in local equilibrium.

The value of c_1 for atmospheric oxygen varies from 0·0037 at 270° K to 0·0069 at 300 °K. An immediate consequence is that shock waves in the atmosphere are fully dispersed if the pressure jump across them is less than $1\cdot0\,\text{lb/ft}^2$ or $1\cdot9\,\text{lb/ft}^2$ at the two temperatures. Since the sonic bang experienced at ground level is that of a reflected shock wave, the bang pressure increases would be twice these values. However, it should be made clear that if the waves are stronger than this the pressure change in the relaxation region of the resulting partially dispersed wave may still dominate the profile.

The relaxation frequency $(\rho_1\phi_1)$ of dry air is $850\,\text{sec}^{-1}$, and this, according to Monk, may increase by a factor of 1000 in hot humid conditions, owing primarily to the increased quantity of water vapour. Since the saturated vapour pressure of water is a sensitive function of the ambient temperature, atmospheric waves in saturated conditions will be wider at lower temperatures.

For a sonic bang of pressure jump $1\,\text{lb/ft}^2$ the shock wave (of half that strength) will have a thickness between 6 m and 6 mm at 300 °K and between 3 m and 30 mm at 270 °K for the range of values of $\rho_1\phi_1$ suggested.

The width of the pressure wave on the ground will be increased by a geometric factor equal to the Mach number of the aircraft defined in terms of the speed of sound at ground level.

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